

ON GEOMETRY OF FRAME BUNDLES

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ABSTRACT. Let (M, g) be a Riemannian manifold, $L(M)$ its frame bundle. We construct new examples of Riemannian metrics, which are obtained from Riemannian metrics on the tangent bundle TM . We compute the Levi-Civita connection and curvatures of these metrics.

1. INTRODUCTION

Let (M, g) be a Riemannian manifold, $L(M)$ its frame bundle. The first example of a Riemannian metric on $L(M)$ was considered by Mok [12]. This metric, called the Sasaki–Mok metric or the diagonal lift g^d of g , was also investigated in [5] and [6]. It is very rigid, for example, $(L(M), g^d)$ is never locally symmetric unless (M, g) is locally Euclidean. Moreover, with respect to the Sasaki–Mok metric vertical and horizontal distributions are orthogonal. A wider and less rigid class of metrics \bar{g} , in which vertical and horizontal distributions are no longer orthogonal, has been recently considered by Kowalski and Sekizawa in the series of papers [9, 10, 11]. These metrics are defined with respect to the decomposition of the vertical distribution \mathcal{V} into $n = \dim M$ subdistributions $\mathcal{V}^1, \dots, \mathcal{V}^n$.

In this short paper we introduce a new class of Riemannian metrics on the frame bundle. We identify distributions \mathcal{V}^i with the vertical distribution in the second tangent bundle TTM . Namely, each map $R_i : L(M) \rightarrow TM$, $R_i(u_1, \dots, u_n) = u_i$ induces a linear isomorphism $R_{i*} : \mathcal{H} \oplus \mathcal{V}^i \rightarrow TTM$, where \mathcal{H} is a horizontal distribution defined by the Levi-Civita connection ∇ on M . By this identification we pull-back the Riemannian metric from TM . We pull-back natural metrics, in the sense of Kowalski and Sekizawa [8], from TM and study the geometry of such Riemannian manifolds. We compute the Levi-Civita connection, the curvature tensor, sectional and scalar curvature.

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2. RIEMANNIAN METRICS ON FRAME BUNDLES

Let (M, g) be a Riemannian manifold. Its frame bundle $L(M)$ consists of pairs (x, u) where $x = \pi_{L(M)}(u) \in M$ and $u = (u_1, \dots, u_n)$ is a basis of a tangent space $T_x M$. We will write u instead of (x, u) . Let (x_1, \dots, x_n) be a local coordinate system on M . Then, for every $i = 1, \dots, n$ we have

$$u_i = \sum_j u_i^j \frac{\partial}{\partial x_j}$$

for some smooth functions u_i^j on $L(M)$. Putting $\alpha_i = x_i \circ \pi_{L(M)}$, (α_i, u_i^j) is a local coordinate system on $L(M)$. Let ω be a connection form of $L(M)$ corresponding to Levi-Civita connection ∇ on M . We have a decomposition of the tangent bundle $TL(M)$ into the *horizontal* and *vertical* distribution:

$$T_u L(M) = \mathcal{H}_u^{L(M)} \oplus \mathcal{V}_u^{L(M)},$$

where $\mathcal{H}^{L(M)} = \ker \omega$ and $\mathcal{V}^{L(M)} = \ker \pi_{L(M)*}$. Let X^h denotes the horizontal lift of a vector field X on M .

Decompose the second tangent bundle TTM into horizontal and vertical part, $T_\zeta T M = \mathcal{H}_\zeta^{TM} \oplus \mathcal{V}_\zeta^{TM}$, with respect to the connection map $K : TTM \rightarrow TM$ and the projection in the tangent bundle $\pi_{TM} : TM \rightarrow M$, see for example [7]. Let $X^{h, TM}$ and $X^{v, TM}$ denote the horizontal and vertical lifts to TTM of a vector field X on M .

For an index $i = 1, \dots, n$ define a map $R_i : L(M) \rightarrow T M$ as follows

$$R_i(u) = u_i, \quad u = (u_1, \dots, u_n) \in L(M).$$

R_i is the right multiplication by a i -th vector of a canonical basis in \mathbb{R}^n .

Proposition 2.1. *The operator R_i has the following properties.*

(1) *We have*

$$R_{i*} X^h = X^{h, TM}.$$

In particular, R_{i} is an isomorphism of $\mathcal{H}^{L(M)}$ and \mathcal{H}^{TM} ,*

(2) *Let \mathcal{V}^i be a linear subspace of $\mathcal{V}^{L(M)}$ spanned by fundamental vertical vectors A^* , where the matrix $A \in \mathfrak{gl}(n)$ has only nonzero i -th column. Then R_{i*} is an isomorphism of \mathcal{V}^i and \mathcal{V}^{TM} , and is zero on \mathcal{V}^j for $j \neq i$. Moreover, there is a decomposition*

$$\mathcal{V}^{L(M)} = \mathcal{V}^1 \oplus \dots \oplus \mathcal{V}^n.$$

Proof. Easy computations left to the reader. □

By Proposition 2.1, we have natural identifications

$$(2.1) \quad \begin{array}{ccccc} \mathcal{H}^{L(M)} & \longleftrightarrow & \mathcal{H}^{TM} & \longleftrightarrow & TM \\ X^h & \longleftrightarrow & X^{h, TM} & \longleftrightarrow & X \end{array}$$

and

$$(2.2) \quad \begin{array}{ccccc} \mathcal{V}^i & \longleftrightarrow & \mathcal{V}^{TM} & \longleftrightarrow & TM \\ X^{v,i} & \longleftrightarrow & X^{v,TM} & \longleftrightarrow & X \end{array}$$

Hence, we have defined the vertical lift $X^{v,i} \in \mathcal{V}^i$ of the vector $X \in TM$ satisfying the property

$$R_{i*}X^{v,i} = X^{v,TM}.$$

Let $c = (c_1, \dots, c_n) \in \mathbb{R}^n$ and $C = (c_{ij})$ be $n \times n$ matrix. We assume that the $(n+1) \times (n+1)$ matrix

$$\bar{C} = \begin{pmatrix} 1 & c \\ c^\top & C \end{pmatrix}$$

is symmetric and positive definite. Let g_{TM} be a Riemannian metric on TM .

Now, we are able to define a new class of Riemannian metrics $\bar{g} = \bar{g}_{\varepsilon, \bar{C}}$ on $L(M)$. Let $F : L(M) \rightarrow TM$ be any smooth function. Put

$$\begin{aligned} \bar{g}(X^h, Y^h)_u &= g_{TM}(X^{h,TM}, Y^{h,TM})_{F(u)}, \\ \bar{g}(X^h, Y^{v,i})_u &= c_i g_{TM}(X^{h,TM}, Y^{v,TM})_{F(u)}, \\ \bar{g}(X^{v,i}, Y^{v,j})_u &= c_{ij} g_{TM}(X^{v,TM}, Y^{v,TM})_{F(u)}. \end{aligned}$$

Fix $u \in L(M)$. Let e_1, \dots, e_n be a basis in $T_x M$, $\pi(u) = x$, such that $(e_1)_{F(u)}^{h, TM}, \dots, (e_n)_{F(u)}^{h, TM}$ is an orthonormal basis in $\mathcal{H}_{F(u)}^{TM}$. Then

$$(2.3) \quad e_1^h, \dots, e_n^h, e_1^{v,1}, \dots, e_n^{v,1}, \dots, e_1^{v,n}, \dots, e_n^{v,n}$$

is a basis in $T_u L(M)$. Let G be a matrix of the Riemannian metric g_{TM} with respect to the basis $e_1^{h, TM}, \dots, e_n^{h, TM}, e_1^{v, TM}, \dots, e_n^{v, TM}$. The fact that \bar{g} is positive definite follows from the following lemma.

Lemma 2.2. *Let*

$$G = \begin{pmatrix} I & g^{hv} \\ g^{vh} & \hat{g} \end{pmatrix}$$

be a positive definite symmetric $2n \times 2n$ block matrix. Then the matrix

$$\bar{G} = \begin{pmatrix} I & c \otimes g^{vh} \\ c^\top \otimes g^{hv} & C \otimes \hat{g} \end{pmatrix}$$

is positive definite.

Proof. It suffices to show that each principal minor \bar{G}_k , $k = 1, \dots, n+n^2$, of \bar{G} is positive. Obviously $\det \bar{G}_k = 1 > 0$ for $k = 1, \dots, n$. Hence we assume $k > n$. Then each minor \bar{G}_k is of the same form as the whole matrix \bar{G} , thus

we will make calculations using matrix \bar{G} . Computing the determinant of the block matrix we get

$$\begin{aligned}\det \bar{G} &= \det(C \otimes \hat{g} - (c^\top \otimes g^{vh})(c \otimes g^{hv})) \\ &= \det(C \otimes \hat{g} - (c^\top c) \otimes (g^{vh} g^{hv})) \\ &= \det((C - c^\top c) \otimes \hat{g} + (c^\top c) \otimes (\hat{g} - g^{vh} g^{hv})).\end{aligned}$$

Since

$$\begin{aligned}\det(C - c^\top c) &= \det \bar{C} > 0, \\ \det \hat{g} &> 0, \\ \det(c^\top c) &\geq 0, \\ \det(\hat{g} - g^{vh} g^{hv}) &= \det G > 0,\end{aligned}$$

it follows that matrices $(C - c^\top c) \otimes \hat{g}$ and $(c^\top c) \otimes (\hat{g} - g^{vh} g^{hv})$ are positive definite. Hence their sum is positive definite. \square

If $\bar{C} = I$ and g_{TM} is the Sasaki metric, then we get Sasaki–Mok metric \bar{g}^d .

Assume now $\bar{C} = I$ and g_{TM} is a natural Riemannian metric on TM [8, 1] such that $g_{TM}(X^h, Y^h) = g(X, Y)$ and distributions $\mathcal{H}^{TM}, \mathcal{V}^{TM}$ are orthogonal. Hence, there are two smooth functions $\alpha, \beta : [0, \infty) \rightarrow \mathbb{R}$ such that

$$\begin{aligned}(2.4) \quad &\bar{g}(X^h, Y^h)_u = g(X, Y), \\ &\bar{g}(X^h, Y^{v,i})_u = 0, \\ &\bar{g}(X^{v,i}, Y^{v,j})_u = 0, \quad i \neq j, \\ &\bar{g}(X^{v,i}, Y^{v,i})_u = \alpha(|F(u)|^2)g(X, Y) \\ &\quad + \beta(|F(u)|^2)g(X, F(u))g(Y, F(u)).\end{aligned}$$

The above Riemannian metric does not "see" the index i of the distribution \mathcal{V}^i . Since all distributions $\mathcal{H}^{L(M)}, \mathcal{V}^1, \dots, \mathcal{V}^n$ are orthogonal, it follows that we may put $F_i(u) = u_i$, that is consider a family of maps F_1, \dots, F_n rather than one map F , in the last condition, to obtain the positive definite bilinear form, hence the Riemannian metric,

$$\begin{aligned}(2.5) \quad &\bar{g}(X^h, Y^h)_u = g(X, Y), \\ &\bar{g}(X^h, Y^{v,i})_u = 0, \\ &\bar{g}(X^{v,i}, Y^{v,j})_u = 0, \quad i \neq j, \\ &\bar{g}(X^{v,i}, Y^{v,i})_u = \alpha(|u_i|^2)g(X, Y) + \beta(|u_i|^2)g(X, u_i)g(Y, u_i).\end{aligned}$$

We will write α_i and β_i instead of $\alpha(|u_i|^2)$ and $\beta(|u_i|^2)$, respectively.

3. GEOMETRY OF \bar{g}

Let (M, g) be a Riemannian manifold, $(L(M), \bar{g})$ its frame bundle equipped with the metric \bar{g} of the form (2.5). Let $\bar{\nabla}$ and \bar{R} denote the Levi-Civita connection and the curvature tensor of \bar{g} , respectively.

We recall the identities concerning Lie bracket of horizontal and vertical vector fields [9]

$$\begin{aligned} [X^h, Y^h]_u &= [X, Y]_u^h - \sum_i (R(X, Y)u_i)^{v,i}, \\ (3.1) \quad [X^h, Y^{v,i}]_u &= (\nabla_X Y)^{v,i}_u, \\ [X^{v,i}, Y^{v,j}]_u &= 0. \end{aligned}$$

Moreover, in the local coordinates, for $X = \sum_i \xi_i \frac{\partial}{\partial x_i}$ we have

$$(3.2) \quad X^h(u_i^j) = - \sum_{a,b} \Gamma_{ab}^j u_i^a \xi_b$$

$$(3.3) \quad X^{v,k}(u_i^j) = \xi_j \delta_{ik}$$

where Γ_{ab}^j are Christoffel's symbols [9].

Proposition 3.1. *Connection $\bar{\nabla}$ satisfies the following relations*

$$\begin{aligned} (\bar{\nabla}_{X^h} Y^h)_u &= (\nabla_X Y)_u^h - \frac{1}{2} \sum_i (R(X, Y)u_i)^{v,i}_u \\ (\bar{\nabla}_{X^h} Y^{v,i})_u &= \frac{\alpha_i}{2} (R(u_i, Y)X)_u^h + (\nabla_X Y)_u^{v,i} \\ (\bar{\nabla}_{X^{v,i}} Y^h)_u &= \frac{\alpha_i}{2} (R(u_i, X)Y)_u^h \\ (\bar{\nabla}_{X^{v,i}} Y^{v,j})_u &= 0 \quad \text{for } i \neq j, \\ (\bar{\nabla}_{X^{v,i}} Y^{v,i})_u &= \frac{\alpha'_i}{\alpha_i} (g(X, u_i)Y^{v,i} + g(Y, u_i)X^{v,i}) \\ &\quad + \left(\frac{\beta'_i \alpha_i - 2\alpha'_i \beta_i}{\alpha_i(\alpha_i + |u_i|^2 \beta_i)} g(X, u_i)g(Y, u_i) + \frac{\beta_i - \alpha'_i}{\alpha_i + |u_i|^2 \beta_i} g(X, Y) \right) U^i, \end{aligned}$$

where $U_u^i = u_i^{v,i}$.

Proof. Follows from the formula for the Levi-Civita connection

$$\begin{aligned} 2\bar{g}(\bar{\nabla}_A B, C) &= A\bar{g}(B, C) + B\bar{g}(A, C) - C\bar{g}(A, B) \\ &\quad + \bar{g}([A, C], B) + \bar{g}([B, C], A) + \bar{g}([A, B], C) \end{aligned}$$

relations (3.1) and the following equalities

$$\begin{aligned} X_u^{v,i}(g(u_i, Y)) &= g(X, Y), \\ X_u^{v,i}(|u_i|^2) &= 2(X, u_i), \\ X_u^h(g(u_i, Y)) &= g(u_i, \nabla_X Y). \end{aligned}$$

□

Before we compute the curvature tensor, we will need some formulas concerning the Levi-Civita connection $\bar{\nabla}$ of certain vector fields.

Lemma 3.2. *The following equalities hold*

$$\begin{aligned}\bar{\nabla}_{X^h} U^i &= 0, \\ \bar{\nabla}_{X^{v,i}} U^j &= 0, \\ \bar{\nabla}_{X^{v,i}} U^i &= \frac{\alpha_i + |u_i|^2 \alpha'_i}{\alpha_i} X^{v,i} + \frac{|u_i|^2 (\alpha_i \beta'_i - \alpha'_i \beta_i) + \alpha_i \beta_i}{\alpha_i (\alpha_i + |u_i|^2 \beta_i)} g(X, u_i) U^i.\end{aligned}$$

and

$$\bar{\nabla}_W (R(u_i, X) Y)^Q = \sum_j W(u_i^j) (R(u_i, X) Y)^Q + \sum_j u_i^j \bar{\nabla}_W (R(\frac{\partial}{\partial x_j}, X) Y)^Q$$

for any $W \in TL(M)$ and Q denoting the horizontal or vertical lift.

Proof. Follows by standard computations in local coordinates. \square

Proposition 3.3. *The curvature tensor \bar{R} satisfies the following relations*

$$\begin{aligned}\bar{R}(X^h, Y^h) Z^h &= (R(X, Y) Z)^h + \frac{1}{2} \sum_i (\nabla_Z R)(X, Y) u_i^{v,i} \\ &\quad - \frac{1}{4} \sum_i \alpha_i (R(u_i, R(Y, Z) u_i) X - R(u_i, R(X, Z) u_i) Y \\ &\quad - 2R(u_i, R(X, Y) u_i) Z)^h, \\ \bar{R}(X^h, Y^h) Z^{v,i} &= (R(X, Y) Z)^{v,i} + \frac{\alpha_i}{2} ((\nabla_X R)(u_i, Z) Y - (\nabla_Y R)(u_i, Z) X)^h \\ &\quad - \frac{\alpha_i}{4} \sum_j (R(X, R(u_i, Z) Y) u_j - R(Y, R(u_i, Z) X) u_j)^{v,j} \\ &\quad + \frac{\alpha'_i}{\alpha_i} g(Z, u_i) (R(X, Y) u_i)^{v,i} - \frac{\beta_i - \alpha'_i}{\alpha_i + |u_i|^2 \beta_i} g(R(X, Y) Z, u_i) U^i, \\ \bar{R}(X^h, Y^{v,i}) Z^h &= \frac{\alpha_i}{2} ((\nabla_X R)(u_i, Y) Z)^h - \frac{1}{2} (R(Z, X) Y)^{v,i} \\ &\quad + \frac{\alpha'_i}{2\alpha_i} g(Y, u_i) (R(X, Z) u_i)^{v,i} - \frac{\alpha_i}{4} \sum_j (R(X, R(u_i, Y) Z) u_j)^{v,j} \\ &\quad - \frac{\beta_i - \alpha'_i}{2(\alpha_i + |u_i|^2 \beta_i)} g(R(X, Z) Y, u_i) U^i, \\ \bar{R}(X^h, Y^{v,i}) Z^{v,j} &= -\frac{\alpha_i \alpha_j}{4} (R(u_i, Y) R(u_j, Z) X)^h \\ \bar{R}(X^h, Y^{v,i}) Z^{v,i} &= \frac{\alpha'_i}{2} (g(Z, u_i) R(u_i, Y) X - g(Y, u_i) R(u_i, Z) X)^h \\ &\quad - \frac{\alpha_i^2}{4} (R(u_i, Y) R(u_i, Z) X)^h - \frac{\alpha_i}{2} (R(Y, Z) X)^h\end{aligned}$$

$$\begin{aligned}
\overline{R}(X^{v,i}, Y^{v,i})Z^h &= \alpha_i(R(X, Y)Z)^h \\
&\quad + \frac{\alpha_i^2}{4}(R(u_i, X)R(u_i, Y)Z - R(u_i, Y)R(u_i, X)Z)^h \\
&\quad + \alpha'_i(g(X, u_i)(R(u_i, Y)Z)^h - g(Y, u_i)(R(u_i, X)Z)^h) \\
\overline{R}(X^{v,i}, Y^{v,j})Z^h &= \frac{\alpha_i\alpha_j}{4}(R(u_i, X)R(u_j, Y)Z - R(u_j, Y)R(u_i, X)Z)^h \\
\overline{R}(X^{v,i}, Y^{v,i})Z^{v,i} &= C_i(g(X, u_i)g(Y, Z) - g(Y, u_i)g(X, Z))U^i \\
&\quad + (A_i g(Y, u_i)g(Z, u_i) + B_i g(Y, Z))X^{v,i} \\
&\quad - (A_i g(X, u_i)g(Z, u_i) + B_i g(X, Z))Y^{v,i} \\
\overline{R}(X^{v,i}, Y^{v,j})Z^{v,k} &= 0 \quad \text{if } \#\{i, j, k\} > 1
\end{aligned}$$

where

$$\begin{aligned}
A_i &= \frac{3(\alpha'_i)^2 - 2\alpha_i\alpha''_i}{\alpha_i^2} + \frac{(\alpha_i\beta'_i - 2\alpha'_i\beta_i)(\alpha_i + |u_i|^2\alpha'_i)}{\alpha_i^2(\alpha_i + |u_i|^2\beta_i)}, \\
B_i &= \frac{\alpha_i\beta_i - 2\alpha_i\alpha'_i - (\alpha'_i)^2|u_i|^2}{\alpha_i(\alpha_i + |u_i|^2\beta_i)}, \\
C_i &= -\frac{2\alpha''_i}{\alpha_i + |u_i|^2\beta_i} + \frac{3\alpha_i(\alpha'_i)^2 + 2(\alpha'_i)^2\beta_i|u_i|^2 + \alpha_i^2\beta'_i - \alpha_i\beta_i^2 + \alpha_i\alpha'_i\beta'_i|u_i|^2}{\alpha_i(\alpha_i + |u_i|^2\beta_i)^2}
\end{aligned}$$

Proof. Follows from the characterization of the Levi-Civita connection $\overline{\nabla}$ and Lemma 3.2. \square

Remark 3.4. Notice that

$$A_i\alpha_i - B\beta_i = C_i(\alpha_i + |u_i|^2\beta_i),$$

which is equivalent to the condition

$$\bar{g}(\bar{R}(X^{v,i}, Y^{v,i})Z^{v,i}, W^{v,i}) = \bar{g}(\bar{R}(Z^{v,i}, W^{v,i})X^{v,i}, Y^{v,i}).$$

Corollary 3.5. *Let X, Y be two orthonormal vectors in the tangent space $T_x M$. Then the scalar curvature \overline{K} of $(L(M), \bar{g})$ and K of (M, g) are related as follows*

$$\begin{aligned}
\overline{K}(X^h, Y^h) &= K(X, Y) - \frac{3}{4} \sum_i \alpha_i |R(X, Y)u_i|^2, \\
\overline{K}(X^h, Y^{v,i}) &= \frac{\alpha_i^2}{4(\alpha_i + \beta_i g(Y, u_i)^2)} |R(u_i, Y)X|^2, \\
\overline{K}(X^{v,i}, Y^{v,i}) &= \frac{A_i(g(X, u_i)^2 + g(Y, u_i)^2) + B_i}{\alpha_i + \beta_i(g(X, u_i)^2 + g(Y, u_i)^2)}, \\
\overline{K}(X^{v,i}, Y^{v,j}) &= 0 \quad \text{for } i \neq j.
\end{aligned}$$

In particular, if (M, g) is of constant sectional curvature κ , then

$$\begin{aligned}\overline{K}(X^h, Y^h) &= \kappa - \frac{3}{4}\kappa^2 \sum_i \alpha_i (g(X, u_i)^2 + g(Y, u_i)^2), \\ \overline{K}(X^h, Y^{v,i}) &= \frac{\kappa^2 \alpha_i^2 g(X, u_i)^2}{4(\alpha_i + \beta_i g(Y, u_i))} \geq 0.\end{aligned}$$

If, moreover, $\sum_i \alpha_i(t_i)t_i < \frac{4}{3\kappa}$ for all $t_i > 0$, then $\overline{K}(X^h, Y^h) > 0$.

Proof. The formula for \overline{K} follows by Proposition 3.3. Since $g(X, u_i)^2 + g(Y, u_i)^2 \leq |u_i|^2$, hence, if (M, g) is of constant sectional curvature and $\sum_i \alpha_i(t_i)t_i < \frac{4}{3\kappa}$, then

$$\overline{K}(X^h, Y^h) \geq \kappa - \frac{3}{4}\kappa^2 \sum_i \alpha_i |u_i|^2 > 0.$$

□

Corollary 3.6. *The scalar curvature \bar{s} of $(L(M), \bar{g})$ at $u \in L(M)$ is of the form*

$$\begin{aligned}\bar{s} &= s - \frac{1}{4} \sum_{i,j,k} \alpha_k |R(e_i, e_j)u_k|^2 \\ &+ \sum_k \left(n(n-1) \frac{B_k}{\alpha_k} + \frac{2(nA_k\alpha_k - B_k\beta_k)}{\alpha_k^2} |u_k|^2 + \frac{(n+3)C_k\beta_k}{\alpha_k^2} |u_k|^4 \right. \\ &\left. + \frac{(n-1)\beta_k(B_k(2\alpha_k + \beta_k) + A_k\alpha_k)}{\alpha_k^2(\alpha_k + |u_k|^2\beta_k)} + \frac{2C_k\beta_k^2}{\alpha_k^2(\alpha_k + |u_k|^2\beta_k)} |u_k|^6 \right),\end{aligned}$$

where s is the scalar curvature of (M, g) and e_1, \dots, e_n is an orthonormal basis in $T_x M$, $\pi_{L(M)}(u) = x$.

Proof. Fix $u \in L(M)$ and let e_1, \dots, e_n be an orthonormal basis in $T_x M$, $\pi_{L(M)}(u) = x$. Then (2.3) forms a basis of $T_u L(M)$. Put

$$\bar{g}_{ij}^k = \bar{g}(e_i^{v,k}, e_j^{v,k}) = \alpha_k \delta_{ij} + \beta_k g(e_i, u_k) g(e_j, u_k).$$

The inverse matrix (\bar{g}_k^{ij}) to (\bar{g}_{ij}^k) is of the form

$$\bar{g}_k^{ij} = \frac{1}{\alpha_k} \delta_{ij} - \frac{\beta_k}{\alpha_k(\alpha_k + |u_k|^2\beta_k)} g(e_i, u_k) g(e_j, u_k).$$

Hence

$$\begin{aligned}\bar{s} &= \sum_{i,j} \bar{g}(\bar{R}(e_i^h, e_j^h) e_i^h, e_j^h) + 2 \sum_{i,j,l,k} \bar{g}_k^{jl} \bar{g}(\bar{R}(e_i^h, e_j^{v,k}) e_l^{v,k}, e_i^h) \\ &+ \sum_{i,j,k,l,p} \bar{g}_k^{ip} \bar{g}_k^{jl} \bar{g}(\bar{R}(e_i^{v,k}, e_j^{v,k}) e_l^{v,k}, e_p^{v,k})\end{aligned}$$

Follows now from Proposition 3.3 and the equality

$$\sum_{i,j} |R(e_i, e_j)u_k|^2 = \sum_{i,j} |R(u_k, e_j)e_i|^2. \quad \square$$

In the case of a Cheeger–Gromoll type metric we have:

Corollary 3.7. *Assume*

$$\alpha_i(t) = \beta_i(t) = \frac{1}{1+t}, \quad t > 0.$$

Then

$$\overline{K}(X^{v,i}, Y^{v,i}) = \frac{-t_i(g(X, u_i)^2 + g(Y, u_i)^2) + t_i^2 + 3t_i + 3}{(1+t_i)^2(1+g(X, u_i)^2 + g(Y, u_i)^2)},$$

where $t_i = |u_i|^2$. In particular, if (M, g) is of constant sectional curvature $0 < \kappa < \frac{4}{3n}$, then sectional curvature \overline{K} is nonnegative.

Proof. We have

$$\sum_i \alpha_i(t_i)t_i = \sum_i \frac{t_i}{1+t_i} < \frac{4}{3\kappa} \quad \text{for all } t_i > 0$$

if and only if $0 < \kappa < \frac{4}{3n}$. Hence, by Corollary 3.5 $\overline{K}(X^h, Y^h) \geq 0$ for $X, Y \in T_x M$ unit and orthogonal. Moreover, $g(X, u_i)^2 + g(Y, u_i)^2 \leq |u_i|^2 = t_i$. Thus

$$\overline{K}(X^{v,i}, Y^{v,i}) \geq \frac{-t_i^2 + t_i^2 + 3t_i + 3}{t_i(1+t_i)^2} = \frac{3}{t_i(t_i+1)} > 0. \quad \square$$

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